Conservation of Linear Momentum (COLM)

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INTRODUCTION

Conservation of Linear Momentum is one of the two most powerful tools available to a collision reconstructionist (the other being Conservation of Energy). But why does it work and how is it applied? This article will attempt to explain the scientific basis and present numeric and graphic techniques to apply conservation of linear momentum.

SCIENTIFIC BASIS

As with many of the tools available to a reconstructionist, COLM is grounded in Newtonian mechanics as described 300 years ago by Sir Isaac, and which have proven to be very accurate predictors of how bodies behave in the world where we live. Newton's Second Law can be stated as:

$$\vec{F} = m \cdot \vec{a} \qquad \textbf{Eq. 1}$$

Where

- $\vec{F}$ = force acting on a body, lb (N)
- m = mass of the body, lb_f*sec^2/ft
- $\vec{a}$ = acceleration experienced by the body, ft/s^2 (m/s^2)
- g = gravity, 32.2 ft/s^2 (9.81m/s^2)

Some of the other terms we will need to define are:

- W = weight, lbf (kg)
- M = Momentum, lbf·sec
- lb_m = pound mass
- lb_f = pound force (often written as simply *lb*)

Weight is the force a mass exerts on the ground when acted on by the acceleration of gravity ($W = mg$), so we can divide each side of this equation by gravity to get mass in terms of weight and gravity:

$$m = \frac{W}{g} \qquad \textbf{Eq. 2}$$

The most common units for mass (in the US) are: $\frac{lbf}{\left(ft / sec^2 \right)}$ which may also be written as (lbm) or sometimes (slugs). We'll see this relationship again.

Acceleration is defined as the change in velocity over a change in time, so we can rewrite Eq. 1 as:

$$\vec{F} = m \cdot \frac{\Delta \vec{V}}{\Delta t} \qquad \textbf{Eq. 3}$$

If we rearrange terms a little bit, we find that:

$$\vec{F} \cdot \Delta t = m \cdot \Delta \vec{V} \qquad \textbf{Eq. 4}$$

So the force multiplied by the time it is acting equals the product of the mass and its change in velocity. The left-hand term in Eq.4 is called the *impulse*, and the right hand term is the change in momentum, since we define momentum as the product of mass and velocity:

$$\vec{M} = m \cdot \vec{V} \qquad \textbf{Eq. 5}$$

Keeping consistent units, with velocity in feet per second, the true units for momentum are:

$$\left(\frac{lbf}{ft/\sec^2}\right)\cdot\left(\frac{ft}{\sec}\right) = \left(lbf\cdot\frac{\sec^2}{ft}\right)\cdot\left(\frac{ft}{\sec}\right) = \left(lbf\cdot\frac{\cancel{\sec^2}}{\cancel{ft}}\right)\cdot\left(\frac{\cancel{ft}}{\cancel{\sec}}\right) = (lbf\cdot\sec)$$

APPLYING COLM TO TWO PARTICLES

Consider two bodies of masses m_1 and m_2 traveling at velocities v_1 and v_2 which collide, then move apart at velocities of v_3 and v_4.

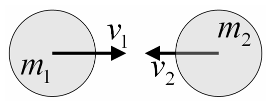


FIGURE 1a: Two balls coming together

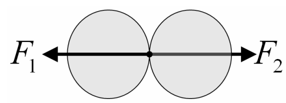


FIGURE 1b: Two balls collide

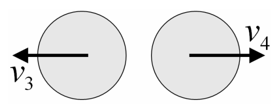


FIGURE 1c: Two balls moving apart

Newton's third law says that the force acting between the two during the collision (in Figure 1b) are equal and opposite:

$F_1 = -F_2$

But Newton's second law says

$F_1 = m_1 a_1$ and $F_2 = m_2 a_2$

So:

$m_1 a_1 = -m_2 a_2$

By the definition of acceleration as the change in velocity divided by the change in time we can write that as:

$$m_1 \cdot \frac{\Delta\vec{V}_1}{\Delta t} = m_2 \cdot \frac{\Delta\vec{V}_2}{\Delta t}$$

The *change-in-time* term cancels out since it is the same for both bodies, and the change in velocity for each body is simply the starting velocity subtracted from the final velocity, so we can write that as:

$$m_1 \cdot (\vec{V}_3 - \vec{V}_1) = -m_2 \cdot (\vec{V}_4 - \vec{V}_2)$$

$$m_1\vec{V}_3 - m_1\vec{V}_1 = -m_2\vec{V}_4 + m_2\vec{V}_2$$

Rearranging, we get the basic conservation of momentum equation:

$$m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_3 + m_2\vec{V}_4 \qquad \textbf{Eq. 6a}$$

The left side of the equation is the total momentum carried IN by the two cars just prior to the collision, while the right side of the equation is the total momentum carried OUT by the two cars just after the collision.

APPLYING COLM TO A CRASH

So momentum changes as a result of a force acting on a body or system over time. In the case of a crash, the "system" is the two vehicles. Let's think about the forces acting on two cars as they collide: they act on each other (internal), the roadway exerts force on the tire contact patches (external), and there are some very very small aerodynamic forces (external).

The *internal* forces act only inside the system - these are the forces of the two vehicles acting on one another. During a typical crash, the internal forces are in the 50,000 lb range.

The *external* forces are those that act on the system from outside – these include the forces the roadway applies to the tire contact patch, and aerodynamic drag forces acting on the bodies of the vehicles. Typical tire forces are less than 2500 lb, and the aerodynamic forces are an order of magnitude smaller still. This means we can usually neglect the tire and aero forces because they are so much smaller than the internal forces.

How about when a car strikes a tree? If we could quantify the forces applied to the tree, we could, in fact, apply COLM, however, the force exerted by the tree on the car can't be measured, calculated, or inferred with any accuracy, so COLM doesn't help us in this case. If we can't quantify the external forces acting on the system, COLM is not much help.

So, to recap, we know several things about crashes which impact our use of COLM:

1. The duration (Δt) is very short (typically 0.080 to 0.120 seconds, or 80 to 120 milliseconds).
2. The duration of a collision is the same for both vehicles involved.
3. The force exerted on Vehicle 1 by Vehicle 2, is equal and opposite the force exerted on Vehicle 2 by Vehicle 1 (From Newton's Third Law).
4. The external tire and aerodynamic forces acting during the impact are much much smaller than the internal forces the two vehicles exert on each other, meaning the tire and aero forces can usually be neglected.

These lead us to the conclusion that during a typical vehicle collision, the total momentum of the vehicles remains essentially the same:

> **Law of Conservation of Linear Momentum**:
> In a system with no external forces acting on it, linear momentum is neither created nor destroyed, it stays the same, or is ***conserved***.

In other words, neglecting the external forces at work during the very short time span of an impact, we can say that the total momentum of the system going into the impact is the same as the total momentum of the system leaving the impact, or "Momentum-IN" equals "Momentum-OUT". The derivation of this was shown as Equation 6, which can also be written as:

$$\vec{M}_{in} = \vec{M}_{out} \qquad \textbf{Eq. 6b}$$

Additionally, as long as there are no significant external forces at work (the bodies under consideration are acting only on each other) they are each exposed to the same force over the same time, so the change of momentum for Vehicle 1 is equal and opposite the change of momentum for Vehicle 2. This can be written as:

$$\Delta\vec{M}_1 = -\Delta\vec{M}_2 \qquad \textbf{Eq. 7}$$

This is important, and we will see it again. The direction of action of the change in momentum is the principal direction of force (PDOF).

Now, we have to note that velocity is a ***vector quantity***: it has magnitude (65 feet per second, for instance) and direction (East, for instance). This is commonly indicated in equations by an arrow or a line over the top of the variable representing the vector quantity. Momentum is a vector quantity, too. Because of this vector-nature, when we add momentums together, we can not simply add their magnitudes, we have to take their directions into account. Examples of how to do this will be presented later in this article.

In order to keep track of pre- and post- impact travel directions, we will use a 360-degree left-handed coordinate system, measuring the angle counter-clockwise from zero, as shown in Figure 1. As long as one is consistent in application throughout an analysis, one can orient the coordinate system in any way, but for simplicity's sake, all cases in this article will orient the coordinates such that vehicle 1 is traveling at 0 degrees immediately pre-impact, along the X-axis. With this coordinate arrangement, vehicle 1's approach angle, α, is always zero, so SIN(α)=0 and COS(α)=1, which simplifies the numeric analysis.

In crash analysis, the "system" is comprised of the vehicles impacting each other. If we have two cars, then the momentum-in can be written as:

$$\vec{M}_{in} = \vec{M}_1 + \vec{M}_2 \qquad \textbf{Eq. 8}$$

where

$\vec{M}_1 =$ pre-impact momentum of Vehicle 1

$\vec{M}_2 =$ pre-impact momentum of Vehicle 2

And the momentum-out can be written as:

$$\vec{M}_{out} = \vec{M}_3 + \vec{M}_4 \qquad \textbf{Eq. 9}$$

where

$\vec{M}_3 =$ post-impact momentum of Vehicle 1

$\vec{M}_4 =$ post-impact momentum of Vehicle 2

Substituting Eq.8 and Eq.9 into Eq.6b, we can write the momentum equation for a two-vehicle impact as:

$$\vec{M}_1 + \vec{M}_2 = \vec{M}_3 + \vec{M}_4 \qquad \textbf{Eq. 10}$$

Using equation 5, we can rewrite equation 10 as:

$$m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_3 + m_2\vec{V}_4 \qquad \textbf{Eq. 11}$$

Substituting Eq. 2 into Eq. 11 for each mass, we get:

$$\frac{W_1}{g}\vec{V}_1 + \frac{W_2}{g}\vec{V}_2 = \frac{W_1}{g}\vec{V}_3 + \frac{W_2}{g}\vec{V}_4 \qquad \textbf{Eq. 12}$$

And we can then multiply both sides of the equation by g, (canceling it out of every term) to get an equation in units of (lbf·ft/sec):

$$W_1\vec{V}_1 + W_2\vec{V}_2 = W_1\vec{V}_3 + W_2\vec{V}_4 \qquad \textbf{Eq. 13}$$

Sometimes it's easier to work with speed in miles per hour. To do this, we can convert all the velocity terms to speed (V ft/sec = 1.467 S mph), to get:

$$(W_1 \cdot 1.467 \cdot \vec{S}_1) + (W_2 \cdot 1.467 \cdot \vec{S}_2) = (W_1 \cdot 1.467 \cdot \vec{S}_3) + (W_2 \cdot 1.467 \cdot \vec{S}_4) \qquad \textbf{Eq. 14}$$

The speed term still has magnitude and direction, so it is a vector, and gets the little over-head arrow to remind us of that fact. Now we can divide every term by 1.467, and rewrite the original momentum equation for two vehicles in units of (lb-mph):

$$W_1\vec{S}_1 + W_2\vec{S}_2 = W_1\vec{S}_3 + W_2\vec{S}_4 \qquad \textbf{Eq. 15}$$

It's worth noting that it doesn't matter which set of units we use for the motion term (feet per second, miles per hour, or even furlongs per fortnight), or the inertia term (mass, weight, or stones), as long as we use the *same* units for every term in the equation. This is why Eqs. 11-15 are all equally valid expressions of Eq.10. For simplicity, units of *lb-mph* will be used for most of this article. If you ever want to use different units, *ft/sec* instead of *mph* for instance, simply replace ALL the speed terms in the momentum equations you are using with a velocity term.

We'll use the following variable conventions:

α (alpha) = Vehicle 1 approach angle	S_1 = Vehicle 1 Pre-impact speed
ψ (psi) = Vehicle 2 approach angle	S_2 = Vehicle 2 Pre-impact speed
θ (theta) = Vehicle 1 departure angle	S_3 = Vehicle 1 departure speed
ϕ (phi) = Vehicle 2 departure angle	S_4 = Vehicle 2 departure sp

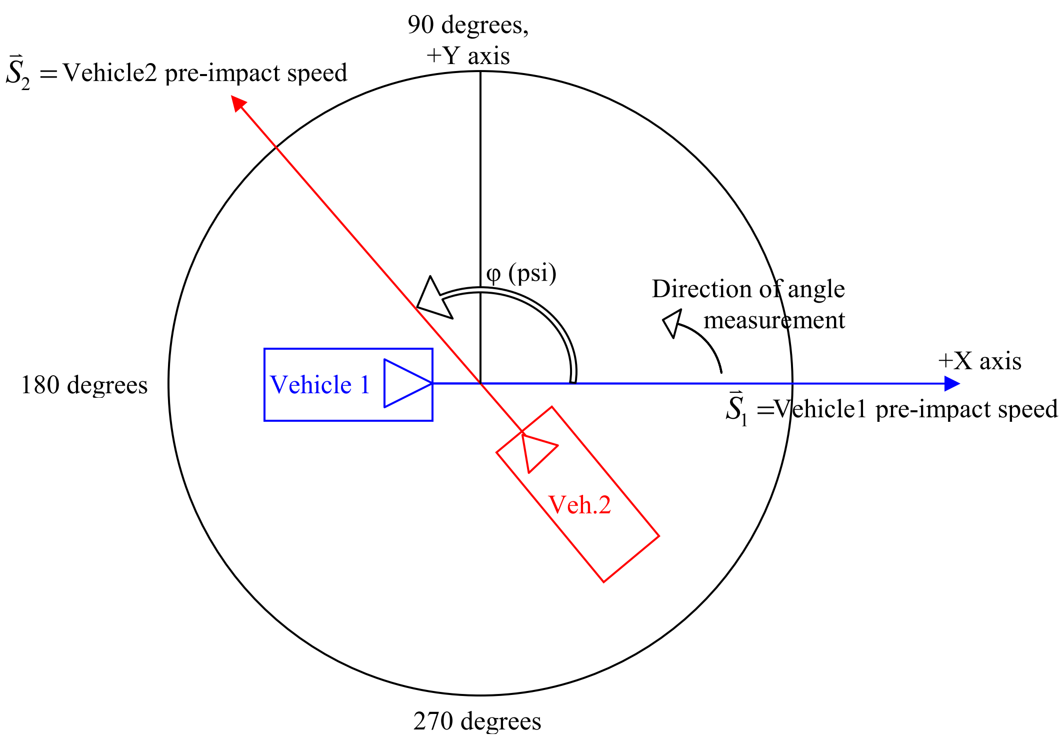


FIGURE 2: The Standard 360° Left-Hand Coordinate System, pre-impact

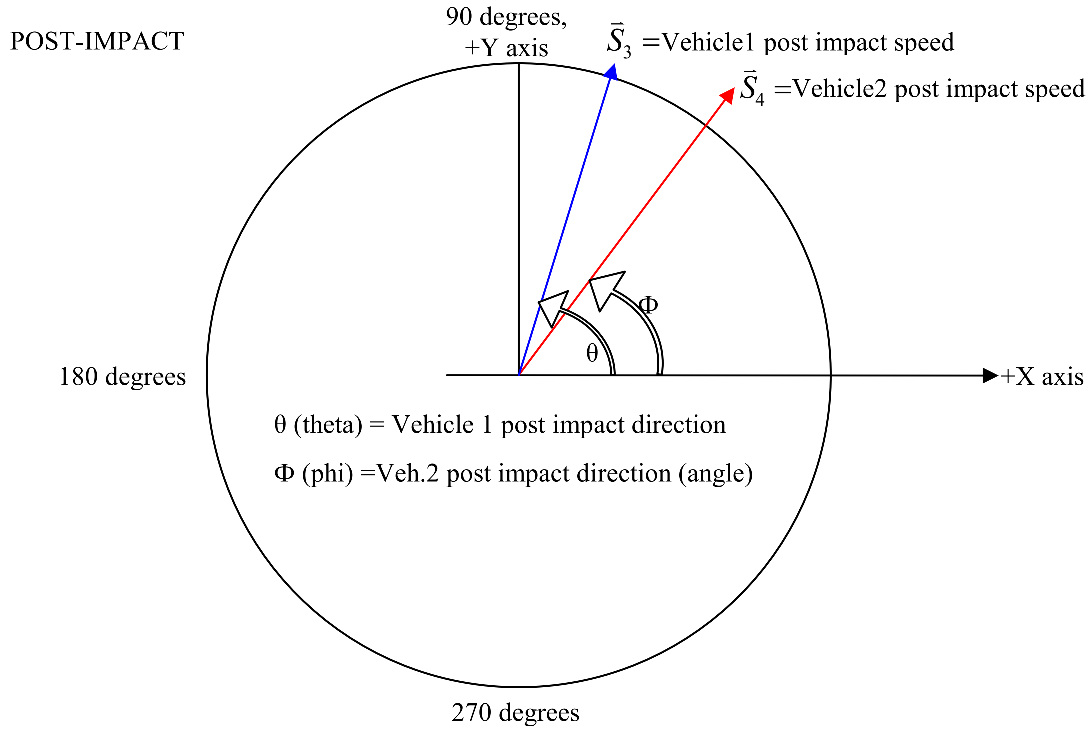


FIGURE 3: The Standard 360° LHCS, post impact

WORKING WITH TWO PERPENDICULAR AXES

Where conservation of linear momentum applies, it can be independently applied to two mutually perpendicular axes, which will be termed here the X and Y axes. By our system definition above, Vehicle 1 will always be traveling along the X-axis at the point of impact. The entire reference frame will simply be rotated to accommodate its travel path orientation at the time of impact. This means we can write two separate and <u>independent</u> equations for linear momentum in the form of Eq.15, one for each axis:

$$W_1\vec{S}_{1X} + W_2\vec{S}_{2X} = W_1\vec{S}_{3X} + W_2\vec{S}_{4X} \qquad \textbf{Eq. 16a}$$

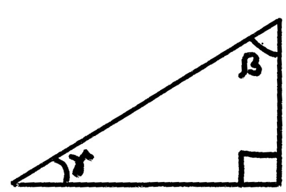
$$W_1\vec{S}_{1Y} + W_2\vec{S}_{2Y} = W_1\vec{S}_{3Y} + W_2\vec{S}_{4Y} \qquad \textbf{Eq. 16b}$$

Usually, we can estimate or measure the vehicle weights and calculate the post impact speeds, leaving us with only the two pre-impact speeds as unknown. With two equations, we can solve for any two unknown values in the equations, and these will usually be the two pre-impact speed terms.

Before starting in on the two axes, a quick review of geometry and resolving a vector into its perpendicular components is in order.

The sum of the angles inside any triangle always equals 180 degrees. In a right triangle, then, the sum of the two non-right angles must be 90. If we can determine one of the angles in a right triangle, we know the other included angle. For instance, using the triangle in Figure 4, we know that $\gamma + \beta = 90$ degrees.

FIGURE 4: A right triangle



If we have parallel lines with a third line crossing through them, as shown in Figure 5, the two indicated angles must be equal. Now, back to vectors. Any vector ($\vec{M}$, for instance) can be broken down into two mutually perpendicular vectors, $\vec{M}_X$ and $\vec{M}_Y$, as shown in Figure 6, by drawing a rectangle using the two perpendicular vectors as two sides. The original vector will cut diagonally across the rectangle. The angles shown are gamma (γ) and beta (β).

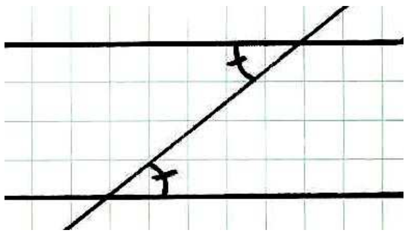


FIGURE 5: Parallel Lines – equal angles

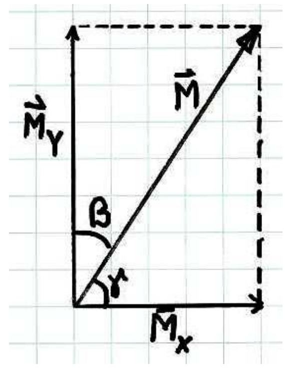


FIGURE 6: Vector components

If we add our two perpendicular vectors head-to-tail we get the original vector, so we can say that the vector M is equal to the vector sum of the two components ($\vec{M} = \vec{M}_X + \vec{M}_Y$):

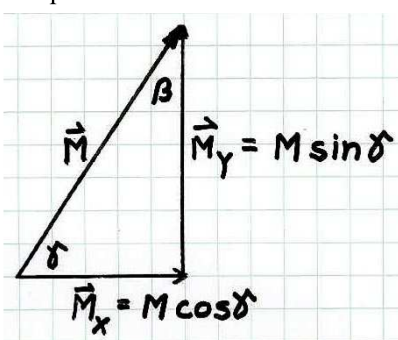


FIGURE 7: Adding vectors

If we have the constituent perpendicular vectors, the magnitude of the vector M can be found using the Pythagorean Theorem for right triangles (the length of the hypotenuse is the square root of the sum of the squares of the sides):

$$M = \sqrt{M_X{}^2 + M_Y{}^2}$$

If we have the length of vector M, and one angle, the lengths of M_x and M_y can be found using the trigonometry relationships shown in Figure 7. (Notice that when working with just the magnitude of the vector, M in this case, the over-bar is not used. The over-bar is only used to indicate a vector.)

Using the terminology conventions denoted earlier, and the relationships shown in Figure 6, we can rewrite Equations 16a and 16b without the vector notation (using just the speed magnitudes). These will be our two most heavily used equations:

$$W_1 S_1 + W_2 S_2 \cos\psi = W_1 S_3 \cos\theta + W_2 S_4 \cos\phi \qquad \textbf{Eq. 17a}$$

$$W_2 S_2 \sin\psi = W_1 S_3 \sin\theta + W_2 S_4 \sin\phi \qquad \textbf{Eq. 17b}$$

Notice that since we have defined the incoming direction of Vehicle 1 as being along the X-axis, it has no momentum in the Y-direction, and its momentum-IN does NOT enter into the Y-axis momentum equation. We can solve ***all*** two-vehicle the linear momentum problems for two vehicles with this pair of equations. How much simpler can it get?

In addition to the speeds, we can evaluate the change in speed (or change in velocity, ΔV) for each vehicle, using these two equations:

$$\Delta S_1 = \sqrt{S_1^{\ 2} + S_3^{\ 2} - 2S_1S_3\cos(\theta)} \qquad \textbf{Eq. 18a}$$

$$\Delta S_2 = \sqrt{S_2^{\ 2} + S_4^{\ 2} - 2S_2S_4\cos(\psi - \phi)} \qquad \textbf{Eq. 18b}$$

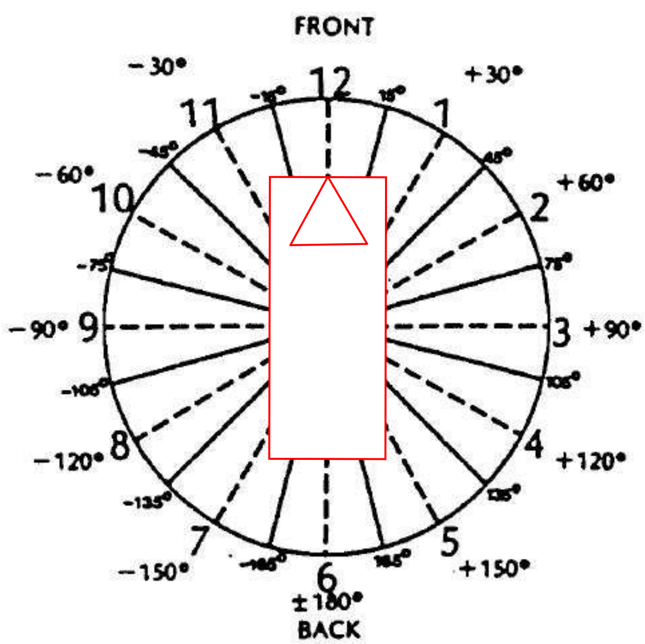
We can also evaluate the Principal Direction of Force (PDOF) for each vehicle, using Equations 19a and 19b for most cases: The $\sin^{-1}$ is called the "*inverse sin*" and is usually accessed on calculators by the <INV><SIN> or <2nd><SIN> keys.

$$PDOF_1 = \sin^{-1}\left(\frac{S_3 \sin\theta}{\Delta S_1}\right) \qquad \textbf{Eq. 19a}$$

$$PDOF_2 = \sin^{-1}\left(\frac{S_4 \sin(\psi - \phi)}{\Delta S_2}\right) \qquad \textbf{Eq. 19b}$$

Remember that the PDOF is measured relative to the direction each vehicle was facing at impact, not its original path of travel. If the car was facing the direction it was going, then 0-degrees is pointed straight back toward the front of the car, 90-degrees indicates a passenger (right) side hit, 180-degrees indicates a rear-end hit, and 270 (or -90) being a driver's side (left) side hit. . The PDOF is NOT anchored to our 360 degree scale. Also, for co-linear crashes, these equations can't differentiate between 0 and 180, since sin(0) = sin(180) = 0. Occasionally, the angle calculated will be off by 90 or 180 degrees. A little common sense and a vector diagram will both go a long way in interpreting PDOF, and we'll see this in action in several examples.

FIGURE 8: PDOF convention (from CRASH3 User's Manual)



USING COLM FOR CO-LINEAR IMPACTS, ONE VEHICLE STATIONARY

The simplest form of collision to evaluate with momentum is a co-linear rear-ender with Vehicle 1 striking a stationary Vehicle 2, as shown in Figure 9. Post impact direction of travel is easy to visualize: the two cars leave the impact in the direction they were facing and traveling, but at speeds different than their original speeds. How much different? That depends on the weights (or masses) of the cars and the impact speed.

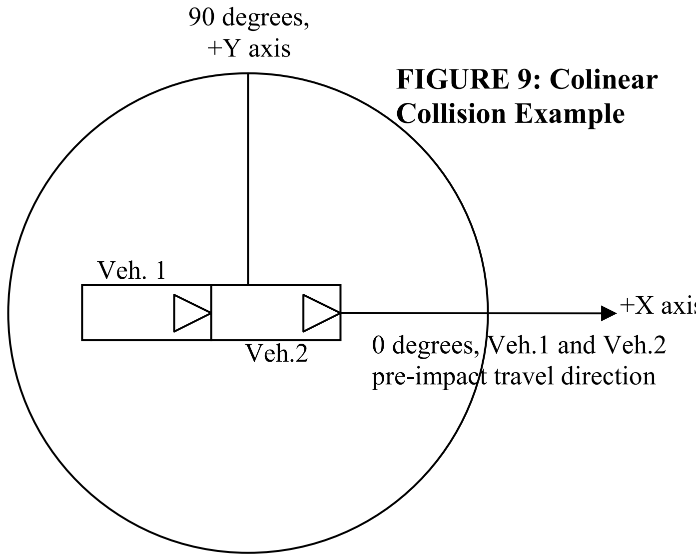


FIGURE 9: Colinear Collision Example

NUMERIC SOLUTION:

Starting with Equation 17a and plugging in our known values (S_2=0, θ =zero, ϕ = zero) we get:

$$W_1S_1 + W_2\underbrace{S_2}_{=0}\cos\psi = W_1S_3\underbrace{\cos(0)}_{=1} + W_2S_4\underbrace{\cos(0)}_{=1}$$

We get the X-axis momentum equation for an inline-impact, with one vehicle stopped before impact:

$$W_1S_1 = W_1S_3 + W_2S_4 \qquad \textbf{Eq.20}$$

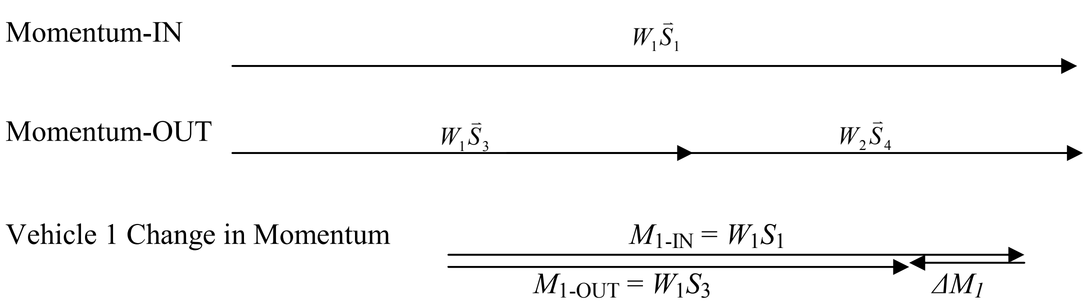
Looking at Equation 17b, though, we find that sin(ψ) = sin(θ) = sin(0) = 0, and all the terms drop out, leaving us with nothing equals nothing:

$$W_2S_2\underbrace{\sin\psi}_{=0} = W_1S_3\underbrace{\sin\theta}_{=0} + W_2S_4\underbrace{\sin\phi}_{=0}$$

Intuitively, this makes sense, though: all the vehicle motion was in one direction (the X-direction) – we shouldn't have anything in the other direction (the Y-direction).

GRAPHIC SOLUTION:

Graphically, we can show the same thing, using arrows to denote the momentum carried by each vehicle. The arrow length is proportional to the vehicle's momentum, and its direction indicates the speed vectors' (and momentum vectors') direction. Using units of *lb-mph*, momentum-IN is shown by two arrows of length W_1S_1 and W_2S_2 added head-to-tail. In this case, though, S_2 is zero, so it doesn't enter into the picture. The length of the two momentum-OUT vectors when so placed must be the same as the length of the single momentum-in vector and in the same direction:



Example 1:
An eastbound pickup truck weighing 3620 pounds (Vehicle 1) strikes a 2150-pound car (Vehicle 2) which is stationary at a stoplight facing east. The pickup's front bumper overrides the sedan's rear bumper and the two vehicles travel together in a straight line to final rest. Using a skid-to-stop analysis, their post-impact speed is determined to have been approximately 28 miles per hour. Find the pickup's pre-impact speed.

Numeric Solution: Since the pickup is the only one in motion pre-impact, we'll call that one Vehicle 1, and orient our X-axis to match its pre-impact travel direction. Now we can make a table with all our information:

	VEHICLE 1	**VEHICLE 2**
Weight	W_1= 3,620 lb	W_2=2,150 lb
Approach Speed	S_1 = ____mph	S_2 = 0 mph
Approach Angle	$\alpha = 0$	$\psi = 0$
	$\cos\alpha = 1$	$\cos\psi = 1$
	$\sin\alpha = 0$	$\sin\psi = 0$
Departure Speed	S_3 = 28 mph	S_4 = 28 mph
Departure Angle	θ = 0 degrees	ϕ = 0 degrees
	$\cos\theta = 1$	$\cos\phi = 1$
	$\sin\theta = 0$	$\sin\phi = 0$

Starting with equation 17a

$$W_1 S_1 + \underbrace{W_2 S_2}_{=0} \cos\psi = W_1 S_3 \underbrace{\cos\theta}_{=1} + W_2 S_4 \underbrace{\cos\phi}_{=1} \qquad \textbf{Eq. 17a}$$

Since S_4=S_3, we can rewrite this as: $W_1 S_1 = (W_1 + W_2) S_3$

and dividing by W_1, we can solve for S_1: $S_1 = \frac{(W_1 + W_2)}{W_1} \cdot S_3$

Substituting the known values gives: $S_1 = \left(\frac{3620 + 2150}{3620}\right) \cdot 28 mph$

With weights measured to three significant figures, this reduces to: $S_1 = (1.59) \cdot 28 mph$
With the post-impact speed calculated to two significant figures, the final solution can only carry two significant figures:

$$S_1 = 44 mph \qquad \textit{NUMERIC ANSWER}$$

Graphic Solution: An alternative approach would be to solve the problem graphically, as shown below. For this example, I will select a scale of 1 inch = 50,000 lbf-mph. Start with what we know: the post-impact momentums. The post impact momentum of vehicle 1 in units of (*lb-mph*) is (3620 *lb* * 28 *mph*) = 101,360 *lb-mph* = 2.03 inches in our newly defined scale. The post impact momentum of vehicle 2 is found to be (2150 *lb* * 28 *mph*) = 60,200 *lb-mph* = 1.20 inches.

Post Impact Momentums:

So, adding them gives us a Momentum-OUT vector that's 3.23 inches long:

$M_{OUT} = M_{IN} = 3.23$ in

Since S_2 =0, ALL the momentum comes from the pickup:

$M_{1\text{-}IN} = W_1S_1 = 3.23$in

At 50,000 lb-mph per inch (the scale we selected earlier) , that converts to 161,500 lbf-mph, so:

W_1S_1 = 161,500 lb-mph

(3620 lb) S_1 =161,500 lb-mph

S_1 =161,500 / 3620 = 45mph *GRAPHIC ANSWER*

Rounding numbers during the graphic conversions produces a result slightly different from the numerical solution.

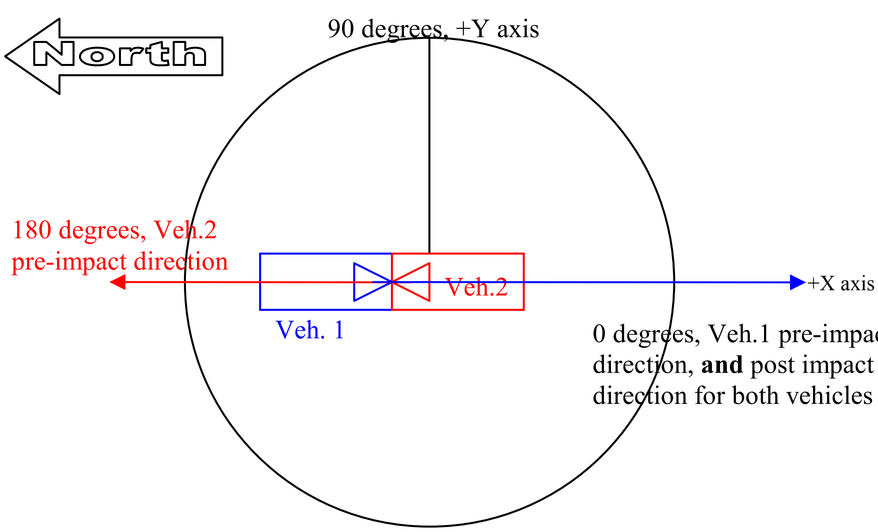
If the two vehicles left the above impact at different speeds, we would start out the same as with the example just worked, but insert both post-impact speeds into Eq.17a, instead of combining them. The rest of the solution would be unchanged.

USING COLM FOR CO-LINEAR IMPACTS, BOTH VEHICLES IN MOTION PREIMPACT

If both vehicles are in motion before the impact, collinear momentum alone is not sufficient to solve for both pre-impact speeds. Momentum will only allow you to calculate the relationship between the two pre-impact speeds. In other words, in order to calculate one vehicle's pre-impact speed, even if both departure speeds can be calculated, the other's speed must be assumed, known (via witnesses or Event Data Recorders, for instance), or calculated by some other means (such as energy analysis).

Example 2:

A southbound Chevrolet Blazer (Vehicle 1, W_1=3,604 pounds) crosses the centerline and strikes a northbound Chrysler Lebaron (Vehicle 2, W_2=3,040 pounds) essentially head on. The Lebaron is pushed south approximately 16 feet, while the Blazer travels approximately 13 feet. Using a skid-to-stop analysis it is determined that the Lebaron's post impact speed was approximately 17 mph, while the Blazer's post impact speed was 15 mph. How fast was the Blazer going before the impact, and what was the Lebaron's speed change?



Numeric Solution

	VEHICLE 1	VEHICLE 2
Weight	W_1= 3,604 lb	W_2=3040 lb
Approach Speed	S_1 = ____mph	S_2 = ____ mph
Approach Angle	α = 0	ψ = 180
	cos α = 1	cos ψ = -1
	sin α = 0	sin ψ =0
Departure Speed	S_3 = 15 mph	S_4 = 17 mph
Departure Angle	θ = 0 degrees	ϕ = 0 degrees
	cos θ = 1	cos ϕ = 1
	sin θ = 0	sin ϕ =0

We know that there's no motion in the Y-axis, so Equation 17b won't do us any good (all the sin terms still equal zero), so starting with equation 17a :

$$W_1S_1 + W_2S_2\cos\psi = W_1S_3\cos\theta + W_2S_4\cos\phi$$

Plugging in the values for the cosine terms gives us:

$$W_1S_1 + W_2S_2\cos\psi = W_1S_3\cos\theta + W_2S_4\cos\phi$$

= -1 = 1 = 1

$$W_1S_1 - W_2S_2 = W_1S_3 + W_2S_4$$

We can rearrange terms to solve for S_1 in terms of S_2:

$$W_1S_1 - W_2S_2 = W_1S_3 + W_2S_4$$

Add W_2S_2 to each side:

$$W_1S_1 - W_2S_2 + W_2S_2 = W_1S_3 + W_2S_4 + W_2S_2$$

$$W_1S_1 = W_1S_3 + W_2S_4 + W_2S_2$$

Then divide by W_1 to solve for S_1 in terms of S_2:

$$S_1 = \frac{W_1S_3 + W_2S_4 + W_2S_2}{W_1}$$

Or we can rewrite this as

$$S_1 = S_3 + \frac{W_2 * S_4}{W_1} + \frac{W_2 * S_2}{W_1}$$

Plugging in the values we do have:

$$S_1 = 15mph + \frac{3040lb * 17mph}{3604lb} + \frac{3040lb * S_2}{3604lb}$$

$$S_1 = 29.3mph + 0.84 * S_2$$

The problem now, of course, is that we have only one equation but two unknowns. We need more information about either S_1 or S_2 to solve for the other one. If a witness comes forward and says they were pacing Vehicle 1 at 45mph, we can rearrange equation 17a as done earlier, but this time solve for S_2 in terms of S_1:

$$S_2 = -S_4 - \frac{W_1 * S_3}{W_2} + \frac{W_1 * S_1}{W_2}$$

$$S_2 = 19mph \qquad \textit{NUMERIC ANSWER}$$

And then the change in speed experienced by the Lebaron can be calculated with

$$\Delta S_2 = \sqrt{S_2{}^2 + S_4{}^2 - 2S_2 S_4 \cos(\psi - \phi)} \qquad \textbf{Eq. 18b}$$

$$\Delta S_2 = \sqrt{19^2 + 17^2 - 2(19)(17)\cos(180 - 0)}$$

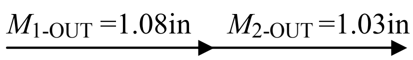
$$\Delta S_2 = 36mph \qquad \textit{NUMERIC ANSWER}$$

Graphic Solution

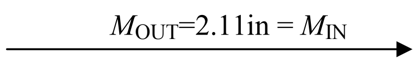
Because of the differences in direction, the graphic representation of this problem is a little more complicated than for Example 1. Using the same 50,000 lb-mph per inch scale as before, we can draw the POST-IMPACT momentum in the same fashion:

$M_{1\text{-OUT}} = W_1 S_3$ = 54,060 lb-mph = 1.081in
$M_{2\text{-OUT}} = W_2 S_4$ = 51,680 lb-mph = 1.033in

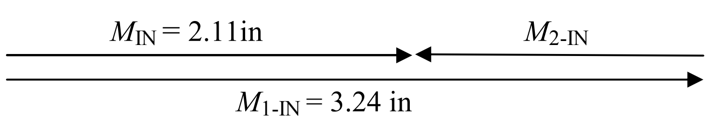
Post Impact Momentums:



So, adding them (HEAD-TO-TAIL) gives us a Momentum-OUT vector that's 2.11 inches:



We know that the head-to-tail addition of the two momentum-IN vectors will equal this momentum-OUT vector, and we know that the two momentum-IN vectors are in opposite directions. What we don't know from this data is how large either of the two vectors is, all we know is their directions. But even this is instructive: We see that Vehicle 1 HAD to be carrying much higher momentum (not necessarily speed) than Vehicle 2 for the combination to have the net result noted earlier. This makes sense intuitively. Using the witness statement that S_1 was 45mph, we find that $W_1 S_1$ = 3604lb * 45mph = 162,180 lb*mph, which is 3.24 inches in our scale, so we can measure the length of $W_2 S_2$:

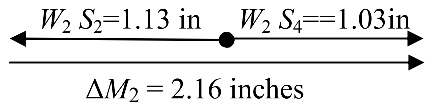


So $M_{2\text{-IN}} = W_2 S_2$ must then be 1.13inches = 56,500 lb*mph, to the left, so

S_2 =56,500 / 3040 = 18 mph *GRAPHIC ANSWER*

Again, the rounding during graphing changes the answer slightly, even with a simple diagram and fairly accurate measurements. This highlights why the graphic method may be good for rough estimates, and for visualizing a wreck, as we'll see later, but a numerical solution can be relied upon for greater accuracy.

What else can we learn from the diagram here? We know that one vector added head-to-tail to another yields a resulting vector, so how about the change in momentum? For vehicle 2, it started out going north at 19mph (with a momentum length of 1.13 inches in our scale), and ended going south at 17mph (with a momentum vector length of 1.03 inches in our scale).



We see that the vector we have to add to M_2 to get M_4 is 2.16 inches long, or 108,000 lb-mph. Remembering that $\Delta M = W\,\Delta S$, so $\Delta S = (\Delta M / W)$, and that this car weighed 3040 lb, we can determine that the change in velocity experienced by the Lebaron (Vehicle 2) was:

$$\Delta S_2 = \Delta M_2 / W_2 = (108{,}000 \text{ lb-mph} / 3040 \text{ lb}) = 35 \text{ mph.}$$ *GRAPHIC ANSWER*

Again, rounding has produced a graphic result slightly different from the numeric result.

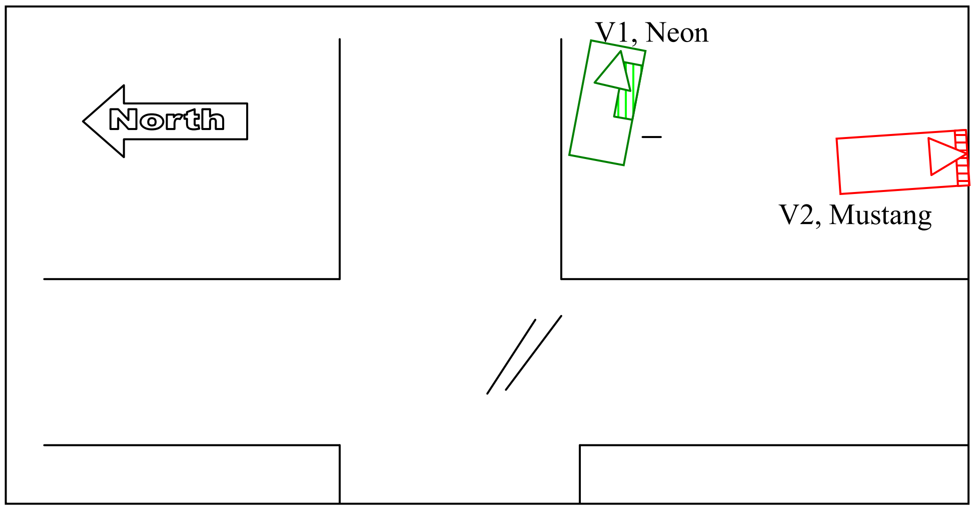
USING COLM FOR NON-CO-LINEAR IMPACTS

What if the two vehicles are not traveling in the same direction (along the same axis)? This is where the 360-degree coordinate system (defined earlier) really shines. By resolving each momentum vector into its X and Y components, we can write the two separate momentum equations described earlier: One for the X-axis and one for the Y-axis. In this way, we can get two equations, which will allow us to solve for two unknowns.

One of the most important aspects of this type of crash is determining the departure angles. This is one place where some people get into trouble – if you simply take the **direction from impact to final rest** you might be right in some cases, but quite often you will be **wrong** because of rollout, or other post-impact phenomena. The angles required for momentum analysis are the direction of the CGs' travel just as the cars separate. Over the years, as the art and science of crash reconstruction has progressed, the understanding of how best to measure the departure angle has changed. As with many analysis techniques, the improved models come with increasing levels of complexity.

In the early days, the direction from impact to final rest was used. We know now that that can lead to gross errors due to rollout. The next commonly used technique was to take the CG's path from first touch to last touch, which really doesn't get us what we seek, which is the post-impact travel (not the during-impact travel). One more recent technique has been to use the line of action from maximum engagement to separation. But in considering the definition of what we're trying to measure (Post-Impact Travel) we see that this is closer than the earlier techniques, but it still isn't quite right. The angle we seek is the direction of the CG's path of travel at the instant the cars stop working on each other. This will be the vehicle's direction of travel prior to any external forces (tire forces) having an effect on it. The best way I know to determine this angle is to plot the vehicle's CG position through the post impact travel using scale cars in a good scale diagram, lining the cars up with scuffs, skids, and gouges. Be careful about simply taking the direction of the gougemarks, as they may have been caused by some part of the car far removed from the CG, and thus not follow the CG's path as the car rotates and translates. There should be an essentially straight line over at least a short distance leaving the impact, before those external tire/roadway forces can redirect the vehicle.

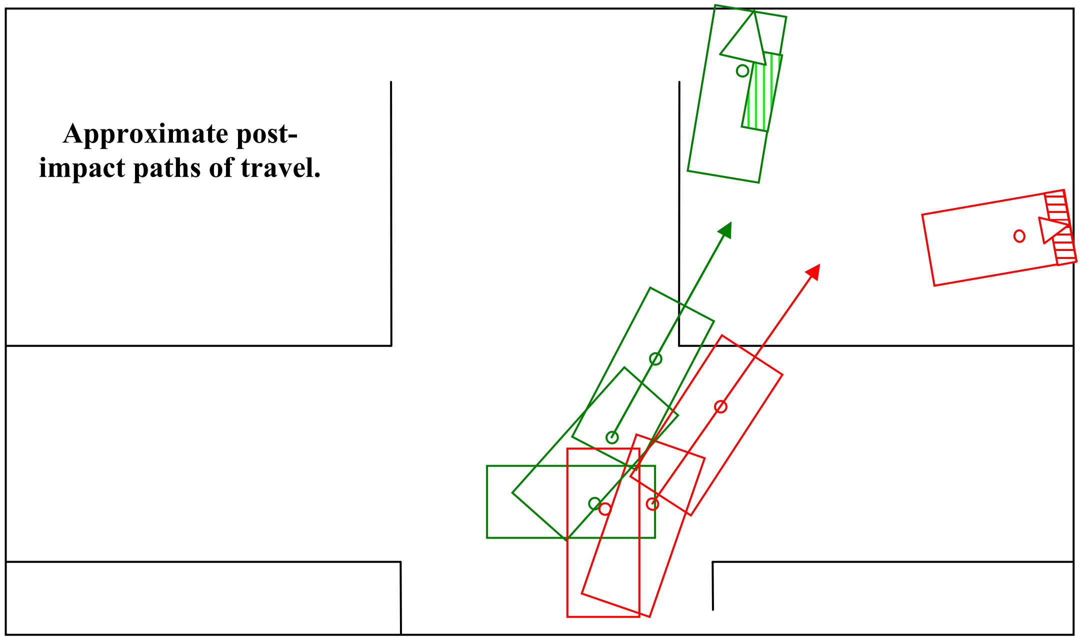
Example 3: Perpendicular approach angles
A Dodge Neon (Vehicle 1, CW=2,513 pounds) carrying one young male occupant (weight~145lb) collides with an eastbound Ford Mustang (Vehicle 2, CW=2,758 pounds) with four male adult occupants (total weight ~ 785lb) at a fully controlled four-way intersection. Two witnesses were in a car following the Mustang, one says the Neon was traveling South, one says it was traveling West. Examining the damage patterns on the two vehicles indicates that the Neon was perpendicular to the Mustang at impact, with the Neon's passenger side heavily impacted. The basic scene diagram is shown below, with fresh gouges in the pavement near the middle of the intersection. Which way was the Neon traveling and how fast were the two cars going?



Directions
With the Mustang headed east, it's only providing momentum in one direction, therefore, we see that the Neon had to supply the momentum headed south to get the cars into that corner of the intersection. So the Neon was NOT going west, it was headed south. The damage, combined with the location of the gouge marks, it is concluded that the Neon was traveling straight through the intersection going south, not turning south from a west-bound lane.

Having selected the Neon as Vehicle 1, we set our X-axis in the Neon's pre-travel direction, or South. Setting scale cars in a scene diagram and plotting the paths of the CGs, it is found that the departure angle for Vehicle 1 is 60 degrees (on our 360-scale) and for Vehicle 2 it's 56 degrees. (Note that the impact to final rest direction would be **wrong**). Since all the people stayed in their respective cars, their weight should be included, as well (Usually this isn't a big deal, but in this case, the people are more than one fifth of the Mustang's total weight, so they should specifically not be neglected!).

Now to the speeds: We've determined the basic direction of travel for both vehicles immediately before impact but we still need the post-impact directions.



Post impact travel for the Neon with one crush-locked front wheel gives a post-impact speed of 19mph. Because the Ford rolled out to final rest, its post-impact speed cannot be conclusively determined from skid-to-stop, but given the solid nature of this hit, it's not unreasonable to assume that the Ford's post impact speed was essentially the same.

Numeric Solution

	VEHICLE 1	**VEHICLE 2**
Weight	W_1= 2,658 lb	W_2=3,543 lb
Approach Speed	S_1 = ____mph	S_2 = ____ mph
Approach Angle	α = 0	ψ = 90
	cos α = 1	cos ψ = 0
	sin α = 0	sin ψ = 1
Departure Speed	S_3 = 19 mph	S_4 = 19 mph
Departure Angle	θ = 60 degrees	ϕ = 55 degrees
	cos θ = 0.5000	cos ϕ = 0.5735
	sin θ = 0.8660	sin ϕ =0.8191

As always, we start with equations 17a and 17b, plug in all our known variables. Because the Neon and Mustang were thoughtful enough to take perpendicular paths to the crash, we can directly solve for S_1 and S_2

$$W_1S_1 + W_2S_2\cos\psi = W_1S_3\cos\theta + W_2S_4\cos\phi \qquad \textbf{Eq. 17a}$$

One term on the left drops out:

$$W_1S_1 + W_2S_2\cos\psi = W_1S_3\cos\theta + W_2S_4\cos\phi$$

=0

Leaving us with this:

$$(2658lb * S_1) = (2658lb * 19mph * 0.50) + (3543lb * 19mph * 0.559)$$

$$(2658lb * S_1) = (25251 + 37630)$$

$$S_1 = (25251 + 37630) lb * mph / 2658 lb$$

$$S_1 = 24 mph \qquad \textit{NUMERIC ANSWER}$$

And taking Eq. 17b we see that here one trig term equals 1, so it disappears

$$W_2 S_2 \underbrace{\sin\psi}_{=1} = W_1 S_3 \sin\theta + W_2 S_4 \sin\phi \qquad \textbf{Eq. 17b}$$

$$3543 lb * S_2 = (2658 lb * 19 mph * 0.8660) + (3543 lb * 19 mph * 0.8191)$$

$$S_2 = (43{,}734 + 55{,}139) / 3543$$

$$S_2 = 28 mph \qquad \textit{NUMERIC ANSWER}$$

The Neon's change in speed is calculated with Eq.18a:

$$\Delta S_1 = \sqrt{24^2 + 19^2 - 2(24)(19)\cos(60 - 0)}$$

$$\Delta S_1 = 21.9 mph \qquad \textit{NUMERIC ANSWER}$$

The Mustang's change is speed is calculated to be 16.5mph

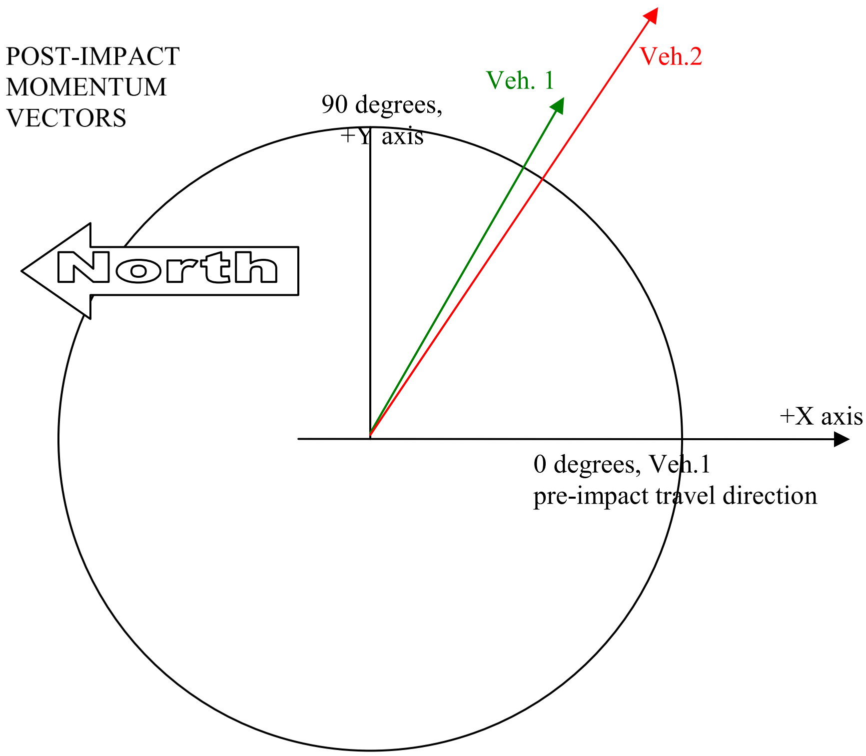
The Neon's PDOF is calculated with Eq.19a:

$$PDOF_1 = \sin^{-1}\left(\frac{S_3 \sin\theta}{\Delta S_1}\right) = \sin^{-1}\left(\frac{19 * 0.8660}{21.9}\right) = 48.7°$$

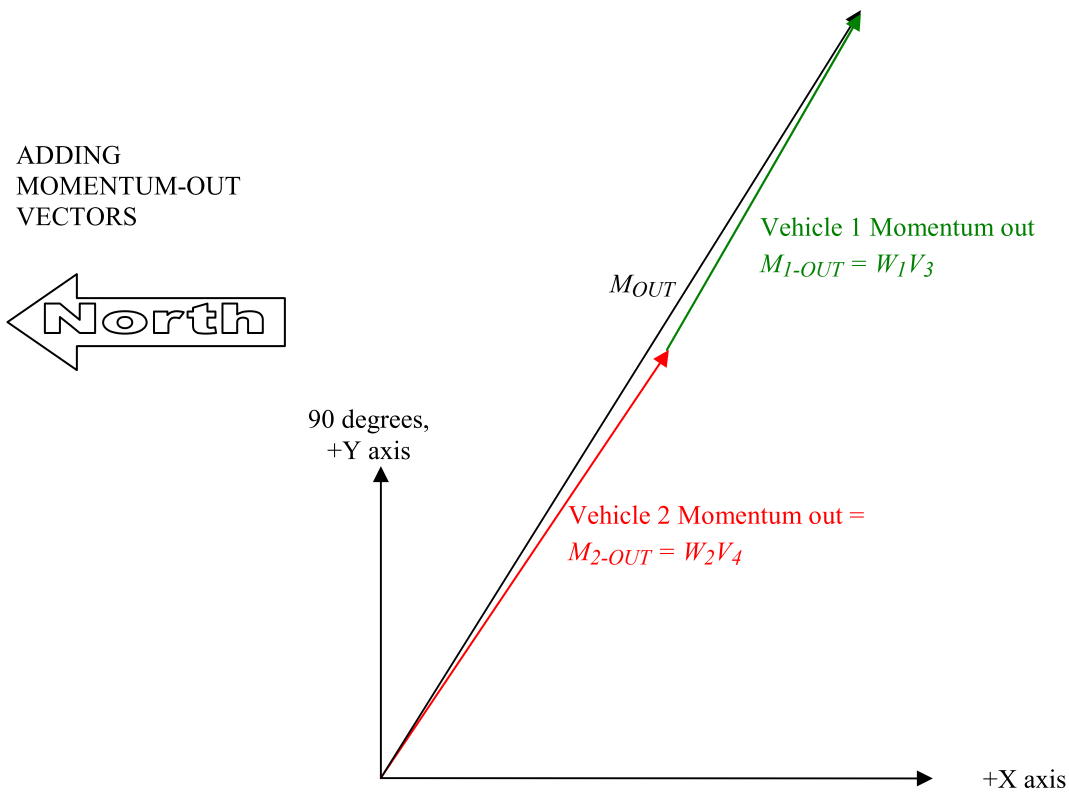
$$PDOF_2 = \sin^{-1}\left(\frac{S_4 \sin(\phi - \psi)}{\Delta S_2}\right) = \sin^{-1}\left(\frac{19 * \sin(-35)}{16.5}\right) = -41.3°$$

Graphic Solution

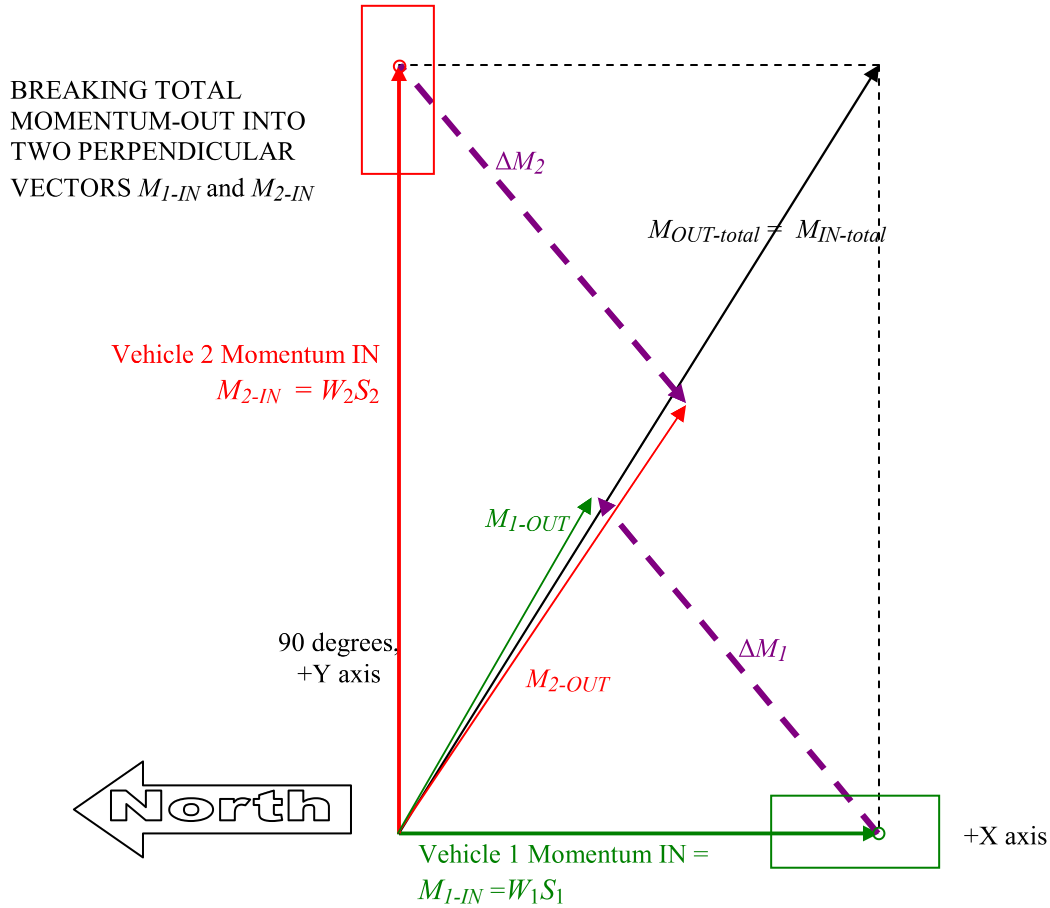
We've got our post impact vector directions from the earlier analysis, we just need their lengths: For V1, the output momentum is (2,658 lb)*19mph = 50,502 lb-mph, and for Vehicle 2, the output momentum is (3,543)*19 = 67,317 lb-mph. *FIND A SCALE:* If we add up the total numerical momentum (117,819) and divide by the available width of our paper (5 inches) we get 23,500 lb-mph per inch. For simplicity, I'll round to the nearest "nice" number, and I'll use a scale of 1 inch=25,000 lb-mph. So our M_3, will be (50,502/25,00) = 2.02 inches long headed 60 degrees, while M_4. will be (67,317/25,000) = 2.69 inches headed 55 degrees from the origin.



Now we add the two momentum-OUT vectors head to tail, to find the total outgoing momentum:



In this case, we know that ALL the X-momentum comes from the Neon, while ALL the Y-Momentum comes from the Mustang, so we draw the parallelogram around the total momentum vector using the X and Y axes as two sides in order to find the two vectors along those axes which add up to the total momentum vector:



The Vehicle 1 (Neon) M_{1-IN} vector is about 2.5 inches long, so:

W_1S_1=2.5in*25,000lb-mph/in = 62,500 lb-mph

S_1=62,500 lb-mph / 2658lb = 24 mph *GRAPHIC ANSWER*

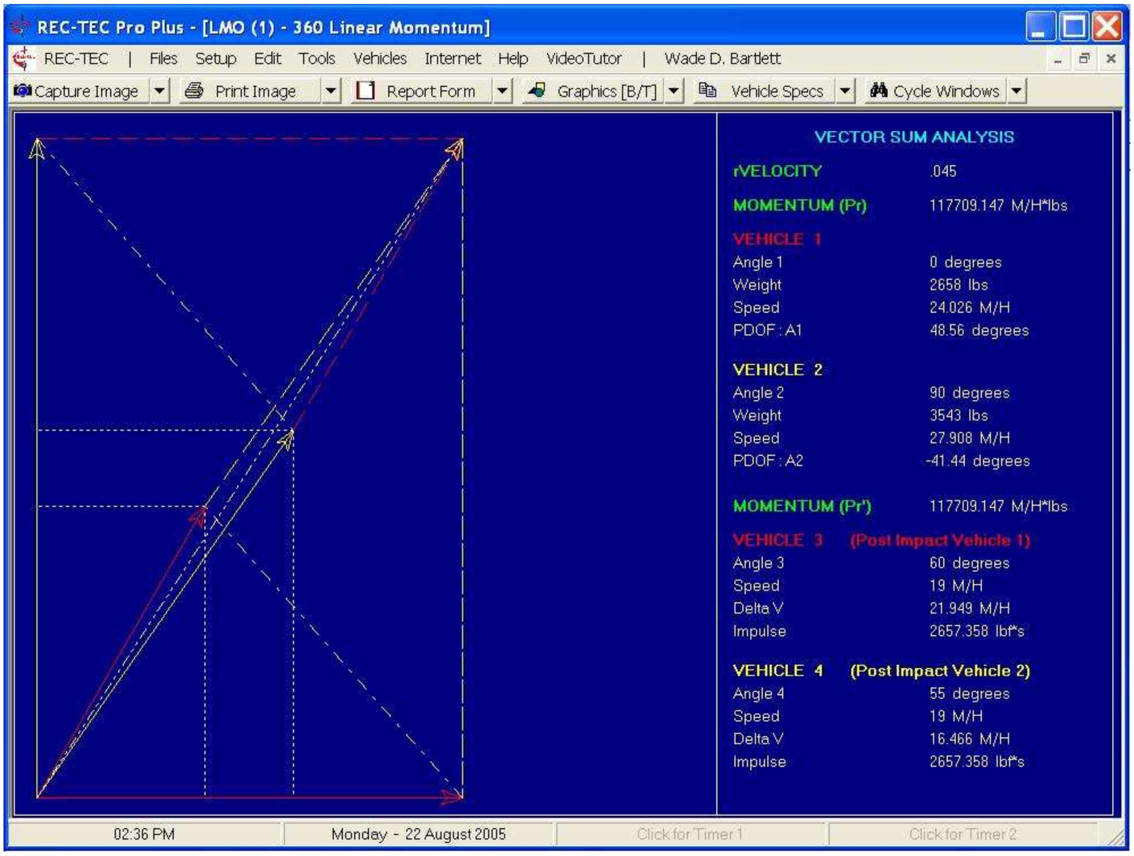
The Vehicle 2 (Mustang) M_{2-IN} vector is about 4 inches long, and running through the same analysis yields:

W_2V_2=4 in * 25,000 lb-mph/in = 100,000 lb-mph

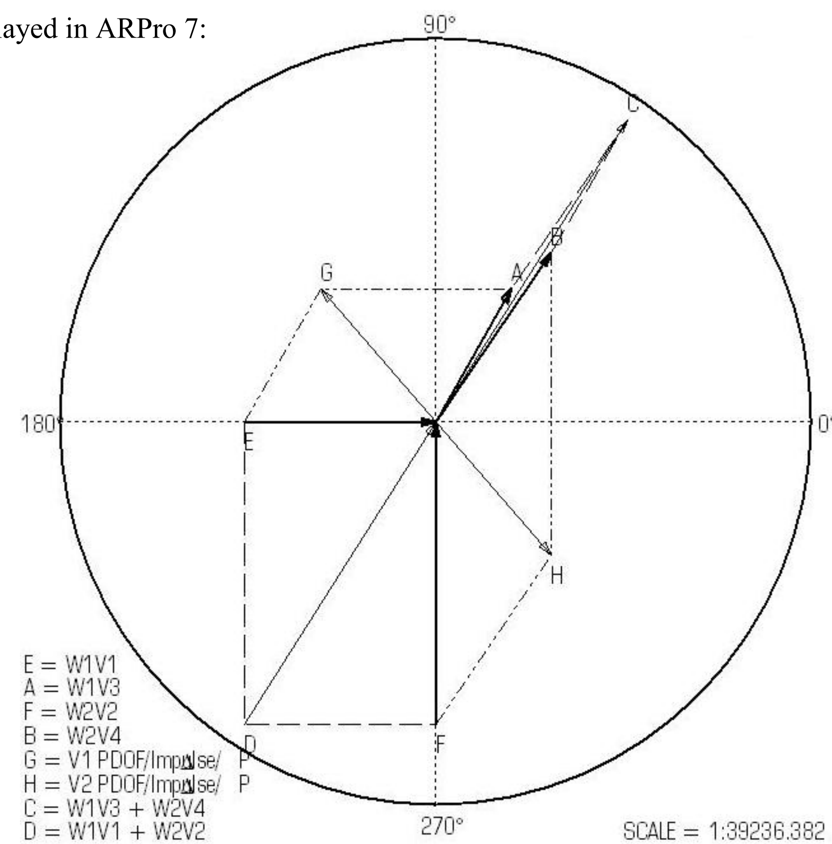
V_2 = 100,000 lb-mph / 3543 lb = 28 mph *GRAPHIC ANSWER*

And the dashed vectors connecting M_1 to M_3 and M_2 to M_4 represent the change in momentum experienced by each vehicle, and they are about 2.28 inches long, or 57,000 lb-mph. We noted earlier that they should be equal and opposite (parallel), and they are oriented at the angle of the principal directions of force. The Neon's speed change from the graphic method is then found to be 57,000/2658 = 21mph, and pointed to the Neon's left and rearward. This time, with slightly better scale resolution, the graphic answer is essentially the same as the numeric answer.

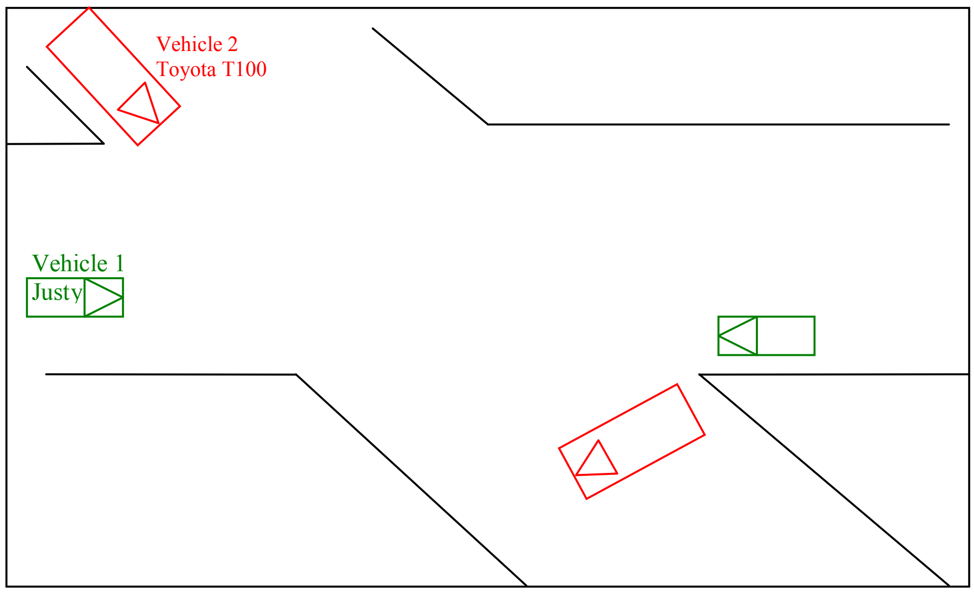
The Neon's PDOF can be measured right off the diagram, and should be a little over 45 degrees.
Vector analysis, as shown by REC-TEC:



Vector Analysis as Displayed in ARPro 7:



Example 4: Non-Perpendicular approach angles
A Toyota T100 pickup carrying no load (CW=3430 lb, Vehicle 2) and two male occupants (weight~350 lb) collides with a Subaru Justy (CW=1805 lb, Vehicle 1) with one female occupant (weight ~ 145 lb) at an oblique intersection on rural roads with speed limits of 40 mph. The impact is non-central, spinning the Justy around 180 degrees, and the Toyota approximately 90 degrees. The pickup has a flashing red light. He claims he stopped at the light, looked both ways, saw noone, proceeded through the intersection, and was struck by the Justy on the left rear. She claims he ran the light, and she didn't have time to stop. The Justy left 24 feet of solid pre-impact skidmarks leading to the impact site. Find the impact speeds for both vehicles and the Justy's change in speed.



Solution
Using a skid-to-stop analysis, the Justy's post impact speed was calculated to be about 28 miles per hour, given the 180 degree rotation and 40 foot travel distance, while the Toyota's post impact speed was determined to be 17 miles per hour.

Numeric Solution

	VEHICLE 1 E on Main	**VEHICLE 2 N on Ash**
Weight	W_1= 1,950 lb	W_2=3,780 lb
Approach Speed	S_1 = ____mph	S_2 = _____ mph
Approach Angle	α = 0	ψ = 317
	cos α = 1	cos ψ = 0.7313
	sin α = 0	sin ψ = -0.6820
Departure Speed	S_3 = 28 mph	S_4 = 17 mph
Departure Angle	θ = 355 degrees	ϕ = 335 degrees
	cos θ = 0.9961	cos ϕ = 0.9063
	sin θ = -0.0871	sin ϕ = -0.4226

$$W_2 S_2 \sin\psi = W_1 S_3 \sin\theta + W_2 S_4 \sin\phi$$ **Eq. 17b**

$$2000 S_2 (1.0) = 3000(30)(0.6428) + 2000(20)(0.4226)$$

$$2000 * S_2 = 57780 + 16904$$

$$S_2 = 74684 / 2000 = 37.3$$

$$-2578 S_2 = (-4{,}759) + (-27{,}182)$$

$$S_2 = (-31{,}941)/(-2578)$$

$$S_2 = 12.4 mph = 12 mph$$ *NUMERIC ANSWER*

And plugging our known values, and the newly found S_2 into 16a:

$$W_1 S_1 + W_2 S_2 \cos\psi = W_1 S_3 \cos\theta + W_2 S_4 \cos\phi$$ **Eq. 17a**

$$3000 S_1 = 3000(30)(0.7660) + 2000(20)(0.906)$$

$$3000 S_1 = (68940) + (36{,}240)$$

$$S_1 = 35.06 mph$$ *NUMERIC ANSWER*

Regarding the Justy's change in speed, we break out equations 18a and 19a:

$$\Delta S_1 = \sqrt{S_1^{\,2} + S_3^{\,2} - 2 S_1 S_3 \cos(\theta)}$$ Eq. 18a

$$\Delta S_1 = \sqrt{(40mph)^2 + (28mph)^2 - 2(40mph)(28mph)(0.9961)}$$

$$\Delta S_1 = 12 mph$$

$$PDOF_1 = \sin^{-1}\left(\frac{S_3 \sin\theta}{\Delta S_1}\right) = \sin^{-1}\left(\frac{28mph * (-0.0871)}{12.5}\right) = -11\deg.$$

And the T100's PDOF calculates to be 55 degrees:

$$PDOF_2 = \sin^{-1}\left(\frac{S_4 \sin(\phi - \psi)}{\Delta S_2}\right) = \sin^{-1}\left(\frac{17 * \sin(18)}{6.4}\right) = 55° *$$

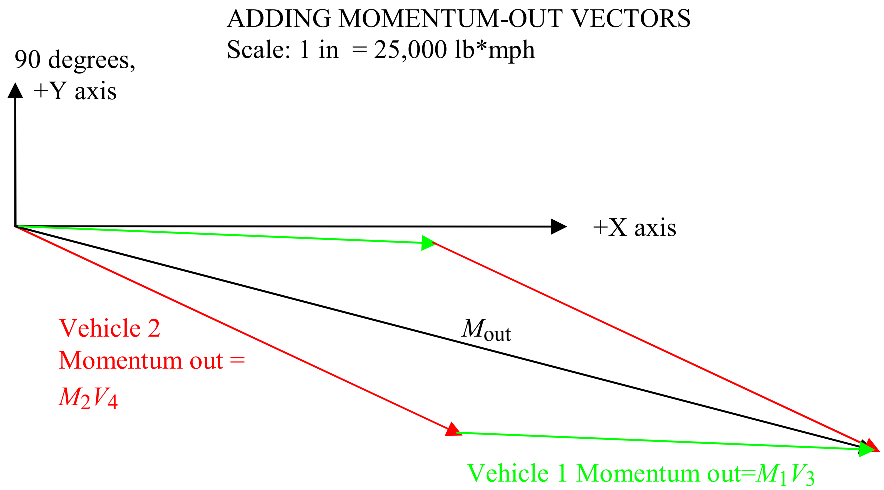
*Watch out now – check out the vector diagram! The true PDOF is (180-55) = 125 degrees or so. Most especially when we cross quadrants, we have to be careful of how the calculated PDOF is really associated with our wreck.

Graphic Solution:

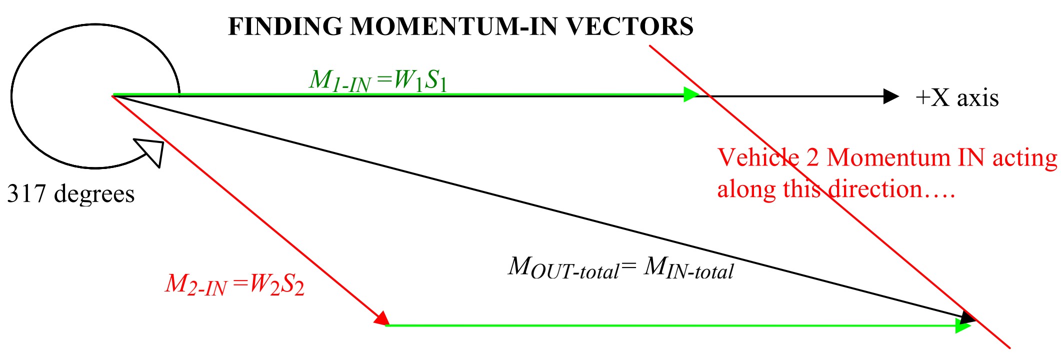
The Vehicle 1 (Justy) M_{out} vector is (1950 *lb* * 28 *mph*) = 54,600 *lb-mph*, which will plot out at 2.184 inches (using a scale of 1 inch = 25,000 lb-mph), at an angle of 355 degrees (or -5degrees).

The Vehicle 2 (T100) M_{out} vector is (3780 *lb* * 17 *mph*) = 64,260 *lb-mph*, which will make it 2.57 inches at 335 degrees.

Putting these two head-to-tail we add them together we see that our total Momentum OUT vector is about 4.5 inches long:



Now, we know the Justy's Momentum-IN vector is along the X-axis (that's the way we set the system up), we just don't know its magnitude yet. We also know that the T100's Momentum-IN vector is along a line of action at 317 degrees. So we can draw a vector at 317 degrees which crosses the end of our M_{out} vector, and we know that where it crosses the X-axis is the end of the Justy's M_{in} vector. Remember that the combination should add up (head-to-tail) to our known total momentum.



Looking at the diagram above, we see that the Justy's momentum-IN vector is about 3 inches long, which with our scale (1in=25,000 lb-mph) means it had 75,000 lb-mph of momentum, and a weight of W_1=1950 lb, so S_1=38 mph *GRAPHIC ANSWER*

And the length of the T100's vector is about 1.9 inches, which equals 47,500 lb-mph, so:

W_2S_2=47,500 lb-mph

S_2 = 47,500 lb-mph / 3780lb

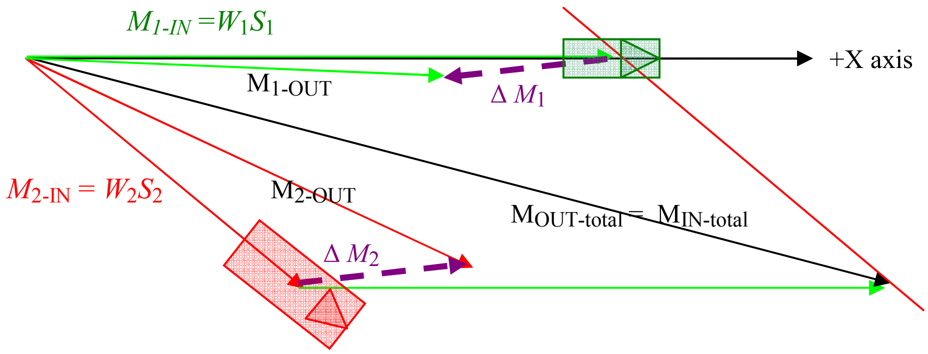
S_2=13mph *GRAPHIC ANSWER*

These numbers compare fairly well with the numeric solution, but the vector diagram provides a more informative visual aid to help understand the events of the crash.

Referring to the vector diagram on the next page, the momentum change vectors are about 0.88 inches long, or 22,141 lb-mph, which yields a speed change of about 11 miles per hour for the

Justy, pointed rearward and slightly to the right. These numbers compare pretty well with the 40 and 12 found numerically above.

FINDING CHANGE IN MOMENTUM & PDOF



SENSITIVITY

An analysis is said to be sensitive to changes in a variable when making small changes in the variable causes big changes in the calculated result. If an analysis is highly sensitive to one (or more) variables, then one may have to allow a wide range in the final answer to be sure that the truth is in there somewhere. To check an analysis' sensitivity, we simply switch one value by a small amount and see how dramatic the effect is. This can often be most easily done with a spreadsheet or an A/R program.

From example 3 for instance, to check the sensitivity of the analysis to the T100's departure angle, we just change that value and redo the analysis with a slightly higher and lower value. The same can be done with the Justy's departure angles. If we have three values for each one, we can make a 9-cell table showing the calculated value with each combination of values:

Calculated Impact Velocity Justy/ T100 for nine departure angle combinations:

		T100 DEPARTURE ANGLE		
		333	335	337
JUSTY DEPARTURE ANGLE	353	37.45 / 13.9	39.06 / 13.12	40.66 / 12.32
	355	38.6 / 13.16	40.2 / 12.38	41.8 / 11.59
	357	39.71 / 12.42	41.32 / 11.64	42.92 / 10.85

Net Difference from Nominal Values (Justy/ T100):

		T100 DEPARTURE ANGLE		
		333	335	337
JUSTY DEPARTURE ANGLE	353	-2.75 / 1.52	-1.14 / 0.74	0.46 / -0.06
	355	-1.6 / 0.78	**0 / 0**	1.6 / -0.79
	357	-0.49 / 0.04	1.1 / -1.12	2.72 / -1.53

So we see that even if the departure angles we used are off by a couple degrees either way, the changes are not huge in this case, but the Justy's result is about twice as sensitive to our angle selections (primarily because it is so much lighter than the T100).

The same sort of sensitivity analysis can be performed for each variable used in the analysis to determine which (if any) have an extraordinary affect on the results. When we have high weight (or momentum) ratios, slight changes in the approach or departure angle of one vehicle (typically the heavier one) can make huge changes in the calculated speeds for the other one.

A WORD ABOUT MOTORCYCLES

Motorcycles present special problems with respect to momentum because (a) they are much lighter than the car, and (b) the rider often gets ejected after some interaction with the car, and goes his own way. The first item means the small changes in the angles selected for the car's pre-and post-impact travel can have an inordinate affect on the calculated motorcycle speed, and the second item means we have to determine the departure angle and departure speed for a third item, namely the rider's body. This extra complication often means momentum is not a viable solution method due to lack of scene data.

If there are three bodies involved in an accident, and we can determine post-impact travel directions and speeds for each of them, we can only solve for two unknowns with momentum. With motorcycles, though, we can usually assume that the rider and motorcycle were traveling at the same speed before the impact, so we only have two unknowns: V1 (car) and V2 (rider/motorcycle).

CONCLUSION:

Conservation of linear momentum is one of the most powerful tools available to the reconstructionist. Numeric solutions can be used to solve for speeds, and graphic methods can be very helpful in not only evaluating the crash speeds, but also for understanding the "bigger picture" of a wreck. If you can create the vector diagram for a crash, you've got the analysis nailed down, and can discuss essentially all the dynamic aspects of the event. This article presented both the numeric and graphic methods, as well as a number of solved examples.

CONTACT

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